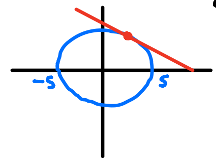
Implicit Differentiation

Motivating example : Find the slope of tangent line to curve $x^2+y^2 = 25$ at $(3,4)$.



Method 1 (Explicit) :

Solve in y and take derivative :

$y = \sqrt{25-x^2}$ (choose + square root as we care about $(3,4)$)

$\frac{dy}{dx} = \frac{1}{2}(25-x^2)^{-\frac{1}{2}} \cdot (-2x)$ (Chain Rule)

$\Rightarrow$ Slope at $(3,4)$ $= \frac{1}{2}(25-3^2)^{-\frac{1}{2}} \cdot (-2 \cdot 3)$

$= \frac{-3}{4}$

Potential Problem : May be very hard to explicitly solve in y

Method 2 (Implicit)

Do not solve for y instead assume it is some function in x.

$y^2 + x^2 = 25$

$\Rightarrow \frac{d}{dx}(y^2 + x^2) = \frac{d}{dx}(25)$

$\Rightarrow \frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) = 0$

$\Rightarrow \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} + 2x = 0$ Chain Rule

$\Rightarrow 2y \cdot \frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

$\Rightarrow$ Slope of tangent at $(3,4) = \frac{-3}{4}$

Drawback : This will give $\frac{dy}{dx}$ in terms of both x and y.

Strategy of Implicit Differentiation (Finding $\frac{dy}{dx}$ when x and y are related by a complicated equation)

1/ Differentiate both sides with respect to x. Use the appropriate rules of differentiation treating y as an unknown function. Simplify until you only have $\frac{dy}{dx}$, x, y terms.

2/ Using algebra rearrange to get $\frac{dy}{dx}$ on one side and x, y terms on the other.

Example

1/ $y^2 + 2y + 1 = x \Rightarrow \frac{dy}{dx} = ?$

$$\frac{d}{dx}(y^2 + 2y + 1) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dx}(y^2) + 2\frac{dy}{dx} + \frac{d}{dx}(1) = 1$$

$$\Rightarrow \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} + 2\frac{dy}{dx} = 1$$

$$\Rightarrow 2y\frac{dy}{dx} + 2\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx}(2y+2) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y+2}$$

2/ $3xy + 2y^2 = 7 \Rightarrow \frac{dy}{dx} = ?$

$$\frac{d}{dx}(3xy + 2y^2) = \frac{d}{dx}(7)$$

$$\Rightarrow \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) = 0$$

$$\Rightarrow \frac{d}{dx}(3x)\,y + 3x\frac{dy}{dx} + \frac{d}{dy}(2y^2)\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow 3y + 3x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(3x + 4y) = -3y \Rightarrow \frac{dy}{dx} = \frac{-3y}{3x+4y}$$

3/ A product is being sold with demand equation $p^2 + 4q^2 = 500$. What is the elasticity when $p = 10$, $q = 10$?

$$\text{Elasticity} = \frac{-p}{q} \cdot \frac{dq}{dp}$$

$$p^2 + 4q^2 = 500 \Rightarrow \frac{d}{dp}(p^2 + 4q^2) = \frac{d}{dp}(500)$$

$$\Rightarrow 2p + \frac{d}{dq}(4q^2) \cdot \frac{dq}{dp} = 0$$

$$\Rightarrow 2p + 8q \frac{dq}{dp} = 0 \Rightarrow \frac{dq}{dp} = \frac{-2p}{8q} = \frac{-p}{4q}$$

$$p = 10, \quad q = 10 \Rightarrow \frac{dq}{dp} = \frac{-10}{40} = \frac{-1}{4}$$

$$\Rightarrow \text{Elasticity} = \frac{-10}{10} \cdot \frac{-1}{4} = \frac{1}{4}.$$